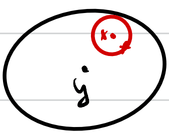
Metric Spaces and Topology
Lecture 3

Open sets. In a metric space (X,d), a subset $U \subseteq X$ is called *open* if it is a union of open balls.

Obs. A set U is open $\Leftrightarrow$ $\forall x \in U$, $\exists$ ball $B_\varepsilon(x) \subseteq U$, i.e. every point in U comes with an entourage.

Proof. $\Rightarrow$. Let $U = \bigcup_{i \in I} B_i$, where each B_i is a ball.

Let's first prove this for one ball $B_r(y)$.

$B_r(y)$


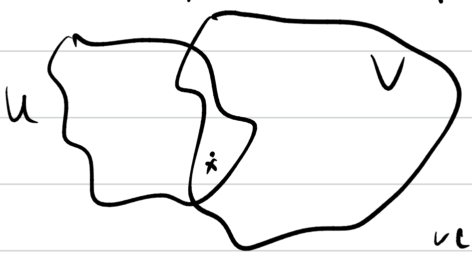
Then $\forall x \in B_r(y)$, $B_{r'}(x) \subseteq B_r(y)$ for $r' := r - d(x,y)$. This holds because of the Δ-inequality.

For a general $U = \bigcup_{i \in I} B_i$, if $x \in U$, then $x \in B_i$ for some $i \in I$, so by the above argument, $B_{r'}(x) \subseteq B_i$ for some $r' > 0$.

$\Leftarrow$. Suppose that every $x \in U$ comes with an entourage $B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$. $\square$

Closure properties. In any metric space (X,d), the class of open sets is closed under (arbitrary) unions and finite intersection.

Proof. The closure under union follows from the definition.
As for finite intersections, it is enough to show for two sets and the rest follows by induction.
Let $U, V \subseteq X$ be open sets. Fix $x \in U \cap V$.



Then $\exists\, r_U$ s.t. $B_{r_U}(x) \subseteq U$
and $\exists\, r_V$ s.t. $B_{r_V}(x) \subseteq V$.
Then, letting $r := \min\{r_U, r_V\}$, we get $B_r(x) \subseteq U \cap V$. $\square$

Examples and nonexamples.

- In $\mathbb{R}$ with usual metric, the following sets are open:
  - open intervals (bounded or not)
  - any union of open intervals
  - **Proposition.** Every open set in $\mathbb{R}$ is a disjoint union of countably many open intervals.

Proof. HW. Hint: Ctblity follows by finding a unique rational in each.

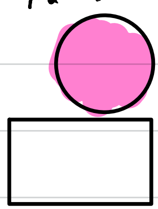
For the existence, prove that each point is contained in a maximal interval.

The following sets are not open in $\mathbb{R}$:

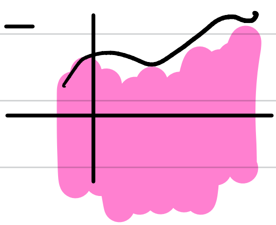
- a singleton $\{x\}$.
- $\{0\} = \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$, this is true for any pt. So even a ctbl intersection of open sets may not be open.

- $X := [0,1)$ with the usual metric. Then
  - $\left(\frac{1}{2}, 1\right)$ is open
  - $\left[0, \frac{1}{2}\right)$ is open (indeed, it is a ball around 0 of radius $\frac{1}{2}$).

- In $\mathbb{R}^2$ with any p-metric, the following are open:
  - open balls = circles
  - open rectangles

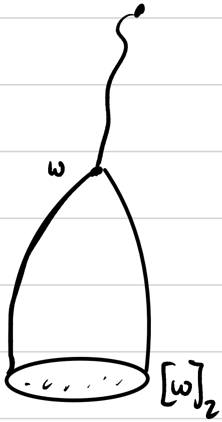


  -

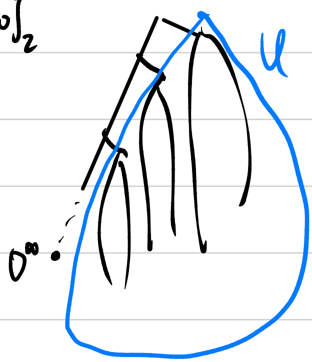


Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Then $\{(x,y) \in \mathbb{R}^2 : y < f(x)\}$. HW

- In the Cantor space $2^{\mathbb{N}}$ (or Baire space $\mathbb{N}^{\mathbb{N}}$), the following are open:
  - the cylinders: for a finite word $w \in 2^{<\mathbb{N}}$ (formally, $A^{<\mathbb{N}} := \bigcup_{n\geq 0} A^n$), the set $[w]_2 := \{wx : x \in 2^{\mathbb{N}}\}$
$= \{y \in 2^{\mathbb{N}} : y = wx\}$
is open since it's an open ball.



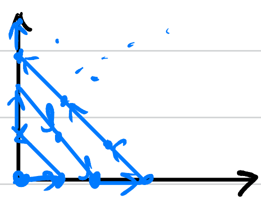
  - $U = [1] \cup [01] \cup [001] \cup [0001] \cup \dots$



  - Proposition. Every open set in $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ is a disjoint union of ctbly many cylinders.

Set theoretic digression. $\mathbb{N} \times \mathbb{N}$ is ctbl.

Proof.



$\square$

Any singleton $\{x\}$ is <u>not</u> open in $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$.

- □ In $X := \{x\}$, this $\{x\}$ is open.
- □ In $X := \{0\} \cup [1,2]$, $\{0\}$ is an open ball of radius 1.

Def. In a metric space (X,d), for a set $Y \subseteq X$, an **interior** point of Y is a point y s.t. $B_\varepsilon(y) \subseteq Y$ for some $\varepsilon > 0$. The **interior of Y**, denoted **$\mathrm{int}(Y)$** or **Y°**, is the set of all interior points.

Note. A set is open if it's equal to its interior.

Obs. $\forall\, Y \subset X$, its interior is open.

Proof. If $B_\varepsilon(x) \in Y$ then every point $y \in B_\varepsilon(x)$ is an interior point, so $\mathrm{int}(Y)$ is the union of all such balls $B_\varepsilon(x)$. □

Obs. In fact, $\mathrm{int}(Y)$ is the $\subseteq$-maximum open subset of Y.

Proof. Let Y' be another open subset of Y. Then each point of Y' is an interior point, hence $Y' \subseteq \mathrm{int}(Y)$. □

<u>Examples.</u>

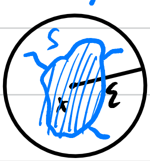
- In $\mathbb{R}$, $\operatorname{int}([0,1)) = (0,1)$.
- In $[0,1)$, $\operatorname{int}([0,\frac{1}{2}]) = [0,\frac{1}{2})$.
- In $\mathbb{R}$, $\operatorname{int}(\mathbb{Q}) = \emptyset$, $\operatorname{int}(\mathbb{R}\setminus\mathbb{Q}) = \emptyset$.
- In $\mathbb{R}$, let $\{q_n\}_{n\in\mathbb{N}}$, let $\mathcal{U} := \bigcup_{n\in\mathbb{N}} (q_n - 2^{-n}, q_n + 2^{-n})$.

  This $\mathcal{U}$ is open and dense since $\mathbb{Q} \subseteq \mathcal{U}$. But $\mathbb{R}\setminus\mathcal{U}$ is nonempty because the "length" of $\mathcal{U}$ is is $\leq \sum_{n\in\mathbb{N}} 2^{-n} = 2$.

  $\operatorname{int}(\mathbb{R}\setminus\mathcal{U}) = \emptyset$ because $\mathcal{U} \supseteq \mathbb{Q}$ but the "length" of $\mathbb{R}\setminus\mathcal{U}$ is ∞.

<u>Terminology.</u> A neighbourhood of a pt $x\in X$ is a set S s.t. $x \in \operatorname{int}(S)$, i.e. $B_\varepsilon(x) \subseteq S$ for some $\varepsilon > 0$.
An open neighbourhood is an open neighbourhood.
An ε-neighbourhood of a point x is a neighbourhood contained in $B_\varepsilon(x)$.

ε-neighborhood



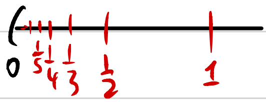
<u>Def.</u> An isolated point in a metric space (X,d) is a point $x\in X$ s.t. $\{x\}$ is open (equivalently, $\{x\} = B_\varepsilon(x)$ for some $\varepsilon > 0$).

<u>Example.</u> 0 in $X := \{0\} \cup (1,2)$.

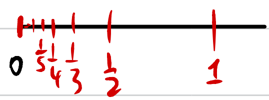
Def. A metric space is **discrete** if every point is isolated.

Examples.

- $\mathbb{N}, \mathbb{Z}^2$ are discrete metric space with the usual metrics.
- $\mathbb{Q}$ is <u>not</u> a discrete metric space, in fact it is **perfect**, i.e. there are no isolated points.
- $X := \{\frac{1}{n} : n \in \mathbb{N}^+\}$ is discrete but there is no minimum distance between two points.



  Indeed, $d(\frac{1}{n}, \frac{1}{n+1}) < \frac{1}{n} \to 0$.
- $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ is <u>not</u> discrete since 0 is not isolated.



Def. We call two metrics d_1, d_2 on X **bi-Lipschitz equivalent** if $\exists\, C, D > 0$ s.t.

$$C d_2 \leq d_1 \leq D d_2.$$ Denote this by $d_1 \simeq d_2$.

We call d_1, d_2 **equivalent** if the metric space (X, d_1) and (X, d_2) have the same open sets. Denote this by $d_1 \sim d_2$.

Obs. $d_1 \sim_L d_2 \Rightarrow d_1 \sim d_2$ but the converse is false, for example: $\mathbb{R}$ and d the usual metric and $d' := \min\{1, d\}$. Then $d \sim d'$ but $d \not\sim_L d'$.

Proof. HW.